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## INVISCID VORTEX MOTIONS IN WEAKLY THREE-DIMENSIONAL BOUNDARY LAYERS AND THEIR RELATION WITH INSTABILITIES IN STRATIFIED SHEAR FLOWS

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# INVISCID VORTEX MOTIONS IN WEAKLY THREE-DIMENSIONAL BOUNDARY LAYERS AND THEIR RELATION WITH INSTABILITIES IN STRATIFIED SHEAR FLOWS.

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## Abstract

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In this report we consider the inviscid instability of three-dimensional boundary-layer flows with a small crossflow over locally concave or convex walls, along with the inviscid instability of stratified shear flows. We show how these two problems are closely related through the forms of their governing equations. A proposed definition of a generalised Richardson number for the neutrally stable inviscid vortex motions is given. Implications of the similarity between the two problems are discussed.

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## §1. Introduction

In a recent paper, Bassom & Hall (1991) have considered vortex instabilities in three-dimensional boundary layers and the relationship between Görtler vortices (Görtler, 1940) and crossflow vortices (Gregory *et al.* 1955). Meanwhile the studies of Hall & Morris (1991) and Hall (1992) have shown that there is also much similarity between the Görtler vortex problem and that for the streamwise vortices in heated-wall boundary layers. Our concern here is with illustrating the close relationship between the inviscid vortex instabilities, considered by Bassom & Hall (1991), and those of stratified shear flow, first considered by Goldstein (1931) and Taylor (1931).

A detailed review of the previous studies related to the Görtler vortex problem can be found in many of the recent papers concerned with this instability. Rather than repeat such material, we refer the reader to the introduction of Bassom & Hall (1991) and references therein. The Reynolds number  $R$  of the flow (to be defined in the next section) will be taken to be large; further, our asymptotic analysis will be restricted to large Görtler numbers  $G$  (also to be defined in the next section). The latter assumption is the more significant, restricting the theory to regions of high (local) wall curvature. Bassom & Hall (1991) have shown that in a 3D boundary layer, the inviscid modes rise in significance to become the most dangerous ones as the amount of crossflow increases. Thus, it is important to investigate the neutral curve bounding these unstable modes. Moreover, a study of neutral vortices is also useful towards the weakly nonlinear analyses of the instability development.

The stability of inviscid stratified shear flows has been considered many times over the past sixty years. This classical problem of hydrodynamic stability theory has been used as a model to consider the stability of atmospheric flows in attempts to explain the phenomena observed in practice. The model incorporates the two competing effects of a potentially unstable shear flow and a stabilising density distribution. The governing equation for the linear instability of such flows is similar to the celebrated Rayleigh's equation, but has the extra buoyancy term which is proportional to a physical parameter generally referred to as the Richardson number.

The rest of this article is divided as follows. In the next section, we outline the derivation of the Bassom-Hall equation governing the stability of inviscid vortices in 3D

boundary layers with weak, *viz.*,  $O(G/Re)^{1/2}$ , crossflows. As shown by Bassom and Hall (1991), the Görtler vortex structures cannot persist if the magnitude of the crossflow exceeds this level. In §3 we discuss the stability properties for both locally concave and convex walls. This is followed by a brief discussion of the numerical methods that were employed in these calculations. In the final section, we begin by outlining the inviscid stability theory for stratified shear flows. We then go on to show their close relationship to the centrifugal-crossflow driven instabilities, or the GSSW (Görtler-Gregory-Stuart-Walker) modes as we propose to call them in §4.

## §2. Inviscid vortex instabilities in 3D boundary layers with weak crossflows.

We consider the flow of a viscous incompressible fluid of kinematic viscosity  $\nu$  over a wall of variable curvature  $r_0^{-1}\chi(x/L)$ . Here  $r_0$  and  $L$  are typical length scales associated with the radius of curvature of the wall and the downstream development of the boundary-layer flow, respectively. Denoting by  $U_\infty$  the free-stream speed sufficiently far from the wall, we define the Reynolds number  $Re$  and the Görtler number  $G$  by

$$Re = \frac{U_\infty L}{\nu} \quad \text{and} \quad G = \frac{2L}{r_0} Re^{1/2}, \quad (2.1a, b)$$

where we restrict our attention to flows where  $Re \gg G \gg 1$ . Next, the dimensionless coordinates

$$X = x/L, \quad Y = Re^{1/2}(y/L), \quad Z = Re^{1/2}(z/L) \quad (2.2a - c)$$

are introduced and the boundary layer velocity and pressure expanded in the form

$$\begin{aligned} \mathbf{u} &= U_\infty \left( \bar{\mathbf{u}} + \epsilon U_0 \mathbf{E} + \dots, Re^{-1/2}[\bar{v} + \epsilon G^{1/2} V_0 \mathbf{E}] + \dots, Re^{-1/2} G^{1/2} [\bar{\lambda} \bar{w} + \epsilon W_0 \mathbf{E}] + \dots \right), \\ p &= \bar{p} + \epsilon Re^{-1} G P_0 \mathbf{E} + \dots, \end{aligned} \quad (2.3a - d)$$

where

$$\mathbf{E} = \exp(iaZ + G^{1/2} \int^x (\beta + \dots) dx), \quad (2.4)$$

and  $\epsilon$  is the nondimensional amplitude of the vortex disturbance. Here  $\bar{u}, \bar{v}, \bar{\lambda} \bar{w}$  and  $\bar{p}$  correspond to the base three-dimensional boundary layer, whilst  $U_0, V_0, W_0$  and  $P_0$  are the respective disturbance functions.

The above scales and expansions lead to the equation

$$(\beta\bar{u} + \bar{\lambda}ia\bar{w})^2(V_{0YY} - a^2V_0) - (\beta\bar{u} + \bar{\lambda}ia\bar{w})(\beta\bar{u}_{YY} + \bar{\lambda}ia\bar{w}_{YY})V_0 + a^2\chi\bar{u}\bar{u}_YV_0 = 0, \quad (2.5)$$

governing the spatial instability of the base flow to steady streamwise inviscid vortices. This equation was first derived by Bassom & Hall (1991). Together with the usual boundary conditions ( $V_0 \rightarrow 0$  as  $Y \rightarrow 0, \infty$ ), it constitutes an eigenproblem for the spatial growth rate  $\beta_r$  (the real part of  $\beta$ ) in terms of the scaled wavenumber  $a$ , the scaled crossflow parameter  $\bar{\lambda}$  and the local wall curvature  $\chi$ . As noted by the above authors, the problem is a localised one and this enables the *magnitude* of  $\chi$  to be scaled out of the problem; thus, we only need to consider the two cases  $\chi = \pm 1$ . The case  $\chi = 1$  corresponds to a wall with locally-concave curvature. Such a curvature is usually necessary for the existence of centrifugal (Taylor–Görtler) instabilities and was the sole case considered by Bassom & Hall (1991). Here we consider both concave- and convex-curved walls, the latter case being synonymous with the choice  $\chi = -1$ . For a discussion concerning the technological relevance of convex curvature, the reader is referred to Bandyopadhyay (1990). The generalisation of equation (2.5) to compressible and hypersonic boundary-layer flows has been studied by Dando (1992) and Fu & Hall (1992), respectively.

Following Bassom & Hall (1991), for a typical base boundary-layer flow, we consider the case of a self-similar Falkner-Skan-Cooke boundary layer with a value of 1/2 for the Hartree parameter. Then,  $\bar{u}$  and  $\bar{w}$  are given by  $\bar{u} = f'(Y)$  and  $\bar{w} = g(Y)$ , where  $f$  and  $g$  satisfy

$$\begin{aligned} f''' + ff'' + \frac{1}{2}(1 - f'^2) &= 0, \quad f(0) = f'(0) = 0, \quad f'(\infty) = 1, \\ g'' + fg' &= 0, \quad g(0) = 0, \quad g(\infty) = 1. \end{aligned} \quad (2.6a, b)$$

The numerical solution of (2.5), plus that of (2.6), is considered in the next section.

### §3. Numerical results of the Bassom–Hall equation for both locally concave and convex walls.

Let us first consider the results for a locally concave curvature ( $\chi = 1$ ). Bassom & Hall (1991) solved equation (2.5) to obtain amplification rates  $\beta$  for several values of  $a$  and

$\bar{\lambda}$ . Their main concern was the effect of crossflow on the growth rates and, hence, they did not attempt to calculate the neutral curve for the inviscid vortices. However, their results indicate that crossflow has a stabilising effect on the inviscid vortices in the sense that, for all  $\bar{\lambda} > 4.69$ , there exists a finite band of wavenumbers where the first mode is stable. In our calculations, we also considered the next ‘most dangerous’ mode and found that it cuts out for a different band of wavenumbers, *i.e.*, the flow is actually unstable for some of the wavenumbers where the first mode is stable. This point was noted independently by Dando (1992) who shows that a consideration of the higher modes is especially important for compressible flows.

In figure 1 we present the neutral curves for the first two modes; it should be noted that unlike most neutral curves, the flow is unstable to its associated mode *outside* of the curve rather than inside. The right hand branch of the neutral curve for each mode has an asymptote of  $a \sim \bar{\lambda}^2$  (with  $\beta \sim \bar{\lambda}^3$ ) as  $\bar{\lambda} \rightarrow \infty$ , as suggested by Bassom and Hall (1991). Therefore, the neutral vortices shrink in their spanwise as well as streamwise wavelengths as the crossflow parameter is increased. It was also found that the critical layer position (where  $\beta_i \bar{u} + \bar{\lambda} a \bar{w} = 0$ ) progressively shifts towards the wall as  $\bar{\lambda} \rightarrow \infty$ , with the vortices aligning themselves with the local flow direction in this region. In other words, the orientation of the vortices is such that  $\beta_i \bar{u}_Y(0) + \bar{\lambda} a \bar{w}_Y(0) = 0$ , which matches with the small wavenumber asymptote in the viscous mode region considered by Bassom and Hall (1991). The left hand branch, on the other hand, tends to a constant spanwise wavenumber  $a = 1.305$ , as  $\bar{\lambda} \rightarrow \infty$ . This limit corresponds to the neutral crossflow vortices that are associated with the inflection point of the directional profile, and were first analyzed by Gregory *et. al.* (1955). It is natural to expect this kind of limiting behaviour since the centrifugal forces will have relatively little effect when the crossflow is increased while keeping the spanwise wavenumber  $a$  to be  $O(1)$ . For the base flow given by (2.6), the orientation of these neutral vortices is given by  $\beta_i / \bar{\lambda} a \approx 0.861$ .

It is also seen from figure 1 that the minimum amount of crossflow required to induce stability in some region of the wavenumber space is smaller in the case of the second mode; however, the band of stable wavenumbers is also much narrower in that case. Another interesting feature of the second-mode neutral curve is the rapid approach of its left-hand

branch to the GSW asymptote. In fact, it is seen from figure 1 that, for all practical purposes, the entire left-hand branch (from  $\bar{\lambda} \approx 4.12$  to  $\bar{\lambda} \rightarrow \infty$ ) may be taken as being equal to the GSW asymptote. Thus, in summary, we find that for locally concave walls ( $\chi = 1$ ), an increase in the amount of crossflow renders stability in a progressively broader range of spanwise wavenumbers; however, the effect of crossflow on the stability properties of different modes is quite non-uniform in the wavenumber space.

Let us now consider the results for locally-convex curvatures ( $\chi = -1$ ). This case was not mentioned by Bassom & Hall (1991), Dando (1992) or Fu & Hall (1992) (however in the latter study the strong curvature of the streamlines of the base flow, *i.e.*, a negative effective Görtler number, is considered). This is probably because the above authors were primarily concerned with the effect of crossflow on the Görtler (1940) instability, rather than the effect of curvature on the crossflow instability of Gregory *et al* (1955). We solved equation (2.5), for  $\chi = -1$ , to obtain amplification rates for several values of  $\bar{\lambda}$  and  $a$ . In figure 2 we present the neutral curve in the  $\bar{\lambda}$ - $a$  plane - note that here, the vortices are unstable *inside* the neutral curve, and stable *outside*. Since convex curvature is stabilizing, there is no steady instability at  $\bar{\lambda} = 0$ , and the minimum amount of crossflow required to induce any steady instability in this case is seen to be  $\bar{\lambda} = 6.42$ . The critical wavenumber corresponding to this minimum crossflow is approximately equal to 0.92. Also notice that in the convex-curvature case, it is the right-hand branch of the neutral curve which asymptotes to the neutral mode of Gregory *et al* as  $\bar{\lambda} \rightarrow \infty$ , whereas the left-hand branch asymptotes to the long-wavelength modes ( $a \rightarrow 0$ ), with the calculations suggesting that  $\bar{\lambda}a \sim -\beta_i \sim \text{constant}$  ( $\approx 3.98$ ). In other words, the critical layer corresponding to the neutral modes along the left-hand-branch moves off to the outer edge of the boundary layer as  $\bar{\lambda} \rightarrow \infty$ . Such a limiting behaviour had also been observed earlier by one of us (MC) in the context of stationary Rayleigh (*i.e.*, pure crossflow) modes in a rotating-disk boundary layer.

In addition to the neutral curve, figure 2 also illustrates the locus of the wavenumber locations corresponding to the maximum amplification rate at any fixed value of the crossflow parameter  $\bar{\lambda}$  (see the dotted curve in figure 2). We have also indicated the numerical values of the spatial amplification rate scaled by the crossflow parameter (*i.e.*,  $\beta_r/\bar{\lambda}$ ) at a

few selected values of  $\bar{\lambda}$ . One may observe that as  $\bar{\lambda}$  increases beyond a value of 15, this ratio approaches fairly rapidly to 0.0336, which is the maximum amplification rate for the crossflow vortices on a surface without any curvature.

It should be clear from the above discussion that, for larger crossflows, the modes of figures 1 and 2 are essentially the crossflow vortices considered first by Gregory *et al* (1955). However, for  $\bar{\lambda} = O(1)$ , their properties are governed crucially both by the crossflow and the wall curvature and hence, we feel that an appropriate term for them, as well as for their counterparts above for a locally concave wall ( $\chi = 1$ ), might possibly be 'Görtler-Gregory-Stuart-Walker' (GGSW) modes. This is how we shall refer to them in the remainder of the paper. We note that equation (2.5), governing the instability properties of these modes, could alternatively be derived by considering the effects of wall curvature on the crossflow modes of Gregory *et al* (1955).

Since the numerical schemes used to obtain the above results also merit discussion by virtue of their novelty, a brief description is now provided. Two types of numerical schemes, both accurate and efficient for this type of calculation, were used. As the neutral inviscid eigenfunctions are singular at the critical layer, integration in  $Y$  has to be along an indented contour in the complex  $Y$  plane. Previous numerical approaches typically used a small indented contour, on which the mean flow properties were computed using a low order Taylor expansion about the critical point. In addition to the loss of accuracy in this procedure, one also needs to shift the position of the contour each time the critical point moves out of the indented portion of the contour (note that in the present problem, the critical layer location varies over a large range). However, by integrating the mean flow equations also on the same contour, (i) one can compute an accurate mean flow on the grid used for stability calculations, and (ii), one can enlarge the indented portion of the contour sufficiently so that the grid only needs to be changed very few times during the calculation of the entire neutral curve (if at all).

One numerical scheme used a fourth order Runge-Kutta scheme and a finite-difference scheme to compute the mean flow and eigenfunction (respectively) along a contour with a large triangular indentation above the real axis into the complex plane, whilst the second approach used was a multi-domain spectral scheme consisting of two segments along the

real axis together with a semi-circular contour of arbitrary size above the real axis. To our knowledge, this is the first multidomain spectral computation involving both real and complex integration domains. The accuracy of the codes was checked against an exact analytic solution that exists for the case of a 2D boundary layer; it was found that the numerical eigenvalues agreed with the analytic ones to six significant figures in the case of the finite-difference scheme and seven significant figures for the spectral scheme.

#### §4. Discussion: the connection with the Taylor–Goldstein equation and stratified shear flows.

In this final section, we illustrate the analogy between the instability problem considered in the previous sections with that for stratified shear flows. The inviscid instability of the latter flows is governed by the so-called Taylor–Goldstein equation (Taylor, 1931; Goldstein, 1931) which we write in the form

$$(\bar{U} - c)^2(\hat{v}_{yy} - \alpha^2 \hat{v}) - (\bar{U} - c)\bar{U}_{yy}\hat{v} + \bar{\beta}_y \bar{U}_c'^2 J \hat{v} / \bar{\beta}_c' = 0, \quad (4.1)$$

where  $\bar{U}$ ,  $\bar{\beta}$  are the nondimensional base shear-flow and the varying density across it, respectively;  $J$  is a parameter measuring the influence of the density gradient relative to the shear of the velocity field; subscript  $c$  indicates that the quantities are evaluated at the critical level,  $y = y_c$ , where  $\bar{U}(y_c) = c$ ;  $\hat{v}$  is the amplitude of the perturbation to the vertical velocity, whilst  $\alpha$  and  $c$  are its wavenumber and wave-speed respectively. Derivations of this equation can be found, for instance, in the papers by Drazin (1958) and Miles (1961).

Usually equation (4.1) is solved on the unbounded domain  $-\infty < y < \infty$ , with a typical case being that of  $\bar{U} = \tanh(y)$  and  $\bar{\beta} = y$  or  $\tanh(y)$ . Together with the boundary conditions  $\hat{v}(\pm\infty) = 0$ , it constitutes an eigenvalue problem for  $c \equiv c(\alpha; J)$  or alternatively  $\alpha = \alpha(c; J)$ . The important parameter  $J$  is known as the (local) *Richardson number* and the eigenvalues are strongly dependent on its value.

Like its close relative, Rayleigh's equation ( $J = 0$ ), the Taylor–Goldstein equation (4.1) is singular at the critical level  $y = y_c$ . However, the singularity is stronger: note that the coefficient of the highest derivative of  $\hat{v}$  has a double zero at  $y = y_c$ . Miles (1961) has derived several theorems concerning solution properties of the Taylor–Goldstein equation;

in particular, (i) that it possesses no unstable solutions for  $J > 1/4$ , and (ii) that the neutral eigenfunctions are proportional to just one of the associated Frobenius solutions,

$$\hat{v} \sim (U - c)^{\frac{1}{2}(1 \pm \sqrt{1-4J})}, \quad (4.2)$$

near the critical level. Note that in atmospheric-flow applications  $J$  is usually considered to be positive (stable stratification) but that in aerodynamical applications it will usually be negative, *i.e.*, as for the boundary-layer flow over a heated plate considered recently by Hall & Morris (1991) and Hall (1992) where buoyancy effects are shown to be strongly destabilising.

Let us now return to equation (2.5) governing inviscid vortex instabilities in 3D boundary layer flows over highly-curved walls. It is immediately clear from a comparison of equations (2.5) and (4.1) that equation (2.5) has (as first deduced and pointed out to the authors by Professor P. Hall: private communication, 1991) essentially the form of the Taylor-Goldstein equation: note that, in particular, both equations are singular at a critical level where the coefficient of the highest derivative has a double zero. Hence the theorems of Miles (1961) are directly applicable to the Bassom-Hall equation (2.5).

It is very easy to derive an analogue of the local Richardson number,  $J$ , for the inviscid vortex (GGSW) instabilities in 3D boundary layers. A quick inspection of equation (2.5), in the neighbourhood of the critical level, indicates that

$$J = -\frac{\alpha^2 \chi \bar{u}_c \bar{u}_c'}{(\beta_i \bar{u}_c' + \bar{\lambda} \alpha \bar{w}_c')^2} \quad (4.3)$$

is the appropriate definition/generalisation. Here  $\beta_i$  represents the imaginary part of  $\beta$ . Immediately, we see that

$$\text{sgn}(J) = -\text{sgn}(\chi), \quad (4.4)$$

*i.e.*, continuing the analogy, we see that convex wall curvature ( $\chi < 0$ ) corresponds to stabilising density stratification; whereas concave wall curvature ( $\chi > 0$ ) corresponds to destabilising density stratification. Thus, (at least) two seemingly unrelated stability problems of classical fluid mechanics are in fact very closely related in a theoretical sense. We remark that the solution properties (for  $J > 0$ ) of the Taylor-Goldstein equation (see

Drazin, 1958) would have motivated a study of the Bassom–Hall equation (2.5) for  $\chi = -1$ , had we not already done so!

Once ‘neutral values’ for the GGSW modes ( $\chi = \pm 1$ ) are calculated, it is very simple to calculate the associated (generalised) Richardson numbers  $J$  using (4.3), as well as the phase jump ( $\phi$  say) of the *linear* neutral eigensolution across the critical level ( $Y_c^+ \rightarrow Y_c^-$ ), which is related to  $J$  via

$$\phi = -\pi(1 \pm \sqrt{1 - 4J})/2. \quad (4.5a)$$

The corresponding behaviour of the eigensolution in the vicinity of the critical level is given by

$$V_0 \sim (Y - Y_c)^{(1 \pm \sqrt{1 - 4J})/2}. \quad (4.5b)$$

Note that the relevant choice of sign in equations (4.5a,b) must, in general, be deduced from an inspection of the numerically calculated eigenfunctions. For locally concave walls ( $J < 0$ ) we found that the minus sign was always appropriate. In other words, the phase shift  $\phi$  is always positive in this case, but the eigenfunction becomes unbounded at the critical level. This appears to be the first example in hydrodynamic stability theory where the vertical velocity eigenfunction does not have a finite norm in the  $\mathcal{L}_\infty$  space. The minus sign is also appropriate when the wall is locally convex ( $\chi = -1$ ,  $J > 0$ ), but only for roughly the right half ( $a > 0.84$ ) portion of the neutral curve in figure 2. The fact that this value is rather close to the critical wavenumber ( $a = 0.92$ ) in figure 2 is purely coincidental. For wavenumbers smaller than 0.84, the ‘+’ sign was found to be appropriate. Of course, since  $J > 0$  in the convex-wall case, the eigenfunction always remains bounded along the neutral curve, irrespective of whether the plus or the minus sign is appropriate in (4.5a,b).

In figure 3, we present a unified plot of the neutral curve in both the concave- and convex-curvature cases in the  $\phi - a$  and, equivalently, in the  $J - a$  plane. Note that the portion from each of the  $\phi$  and  $J$  curves with  $a < 1.305$  corresponds to  $\chi = -1$ , with the remaining one ( $a > 1.305$ ) being related to the first mode for  $\chi = 1$ . The link between these two portions, *viz.*,  $a = 1.305$ , corresponds to the neutral wavenumber of the crossflow vortices, *i.e.*, the limit  $\bar{\lambda} \rightarrow \infty$ , considered by Gregory *et al* in 1955. As alluded to previously, it is not surprising that the curvature will have little influence on the solutions of the Bassom–Hall equation in this particular limit, and accordingly, one finds that both

curves are continuous at this wavenumber. What is perhaps more interesting is that the *slopes* of the graphs are also continuous there. This indicates that two seemingly physically different problems  $\chi = \pm 1$  are in fact 'two halves' of the same problem in a mathematical sense, being connected through the  $\lambda \rightarrow \infty$  limit (or, alternatively, through  $a = 1.305$ ; see figures 1 and 2). It also appears to confirm the accuracy of our computed numerical solutions.

Note that the shape of the  $J$  vs.  $a$  curve in figure 3 is remarkably similar to that appearing in Drazin's (1958) figure 2. Moreover,  $J \leq 1/4$  for the GGSW modes, in full agreement with a theorem of Miles (1961) for the instabilities of stratified shear flows. One may also observe that the value of  $J$  asymptotes to a finite constant (*viz.*,  $-2$ ) as the wavenumber becomes large. This is unlike the usual stratified flow problems (Drazin 1958, Hazel 1972) where, typically,  $J \rightarrow -\infty$  as  $a \rightarrow \infty$ . We believe that the limited range of possible Richardson numbers for the neutral GGSW modes is the result of the base flow being wall bounded, as against the free shear flows studied usually in the stratified flow context. We found that the second mode for  $\chi = 1$  has  $J$  values ranging from  $-4$  to  $-7$ , although, unfortunately, the asymptotic nature of these boundaries could not be established due to the considerable numerical difficulties encountered in higher-mode computations.

Finally, one may see from figure 3 that the phase shift  $\phi$  takes *all* values between  $-\pi$  and  $\pi$ , in contrast to many other linear problems where the phase shift only takes the values  $-\pi$  or  $0$  depending on the presence and location of so-called inflection points. In the concave-wall case, the magnitude of the phase shift and, hence, the degree of singularity near  $Y = Y_c$  increases monotonically as one moves from the GSW asymptote ( $\phi = 0$ ) to the wall-mode asymptote ( $\phi = \pi$ ) along the neutral curve in figure 1. On the other hand, the phase shift in the case of a convex-wall is found to decrease from  $0$  to  $-\pi$  as one moves from the GSW asymptote to the small-wavenumber asymptote in figure 2. Moreover, the degree of singularity in the neutral eigenfunction increases inward from both asymptotes, achieving a  $(Y - Y_c)^{1/2}$  type behaviour near  $a = 0.84$ , where  $J = 1/4$ . Since  $J = 1/4$  does *not* correspond to the minimum crossflow ( $\bar{\lambda} = 6.42$ ) as mentioned previously, this particular value of the Richardson number does not have the kind of physical significance

in the present problem which it has in the stratified-flow context.

## §5. Conclusion.

We now finish with a few remarks in conclusion. Bassom & Hall (1991) have shown that the inviscid instability of weakly three-dimensional boundary layers over concave curved walls is very closely connected to the crossflow instability of Gregory *et al* (1955). We have extended their study to convex-curved walls and have shown that all these instabilities of boundary-layer flows are, moreover, related mathematically to instabilities of stratified shear flows. Physically, the common feature between these two classes of instabilities is the presence of an inviscid body force which, along with the inertial effects associated with the shear flow, can have a profound impact on the stability of the flow. This body force corresponds to the centrifugal force due to surface curvature in the former case, whilst it is induced by the density stratification in the case of the latter class of flows. The effects of this body force relative to that of the shear-flow instability can always be characterized in terms of some suitable analogue of the Richardson number. The proposed form for this generalized Richardson number for the GGSW modes is given by equation (4.3). It would be very simple indeed to 'doubly' generalise this definition to compressible boundary-layer flows and boundary-layer flows over heated plates.

The closeness between the two problems also suggests that ideas and theories from the many studies of stratified shear flows (many of which, themselves, are adaptations from the corresponding ideas and theories for homogeneous shear flows) can be applied to future studies concerning the GGSW modes, involving, for instance, their weakly and/or fully nonlinear evolution subsequent to the linear growth stage. Another aspect that may warrant further investigation as far as the GGSW modes are concerned is the answer to the question whether any of these modes are 'absolutely unstable,' since in the corresponding study for stratified shear flows, Lin & Pierrehumbert (1986) do find regions of 'absolute' instability (to date, Görtler vortex instabilities have generally been found, as well as observed, to be 'convective' in nature).

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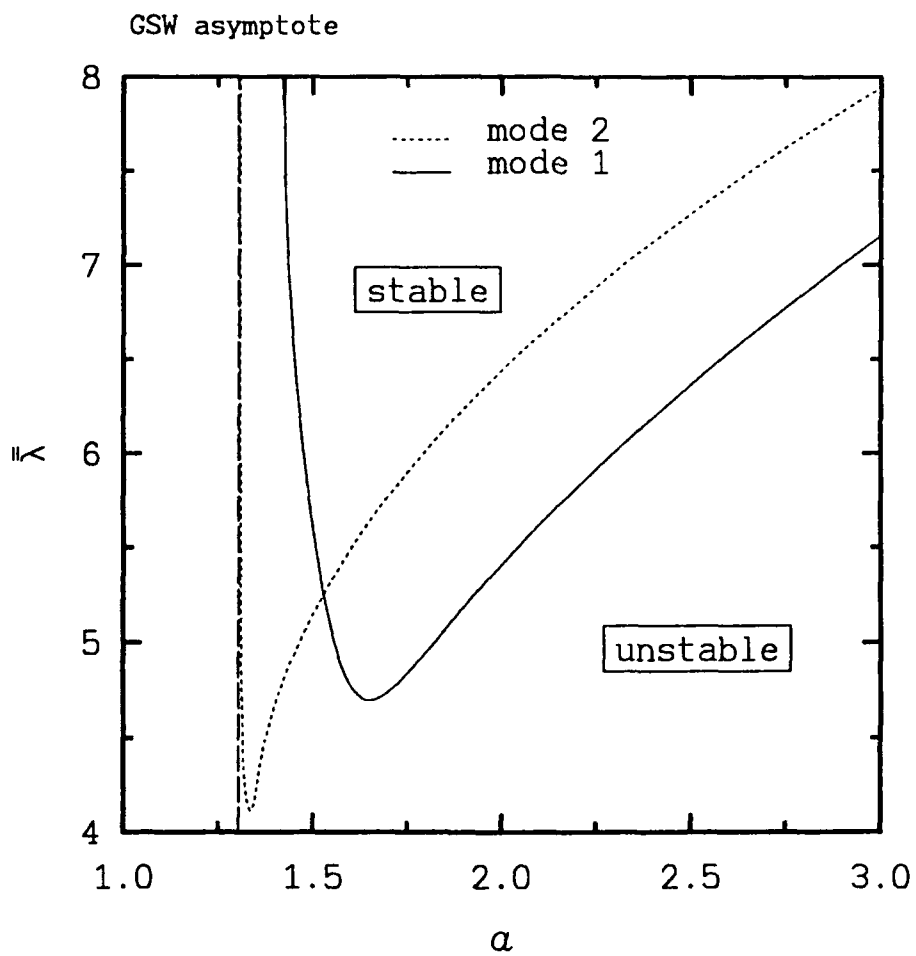


Fig. 1 The neutral curve for the first two modes of stationary inviscid vortices in the flow over a concave wall. Here  $\bar{\Lambda}$  denotes the scaled crossflow parameter, while  $a$  is the dimensionless wavenumber in the spanwise direction.

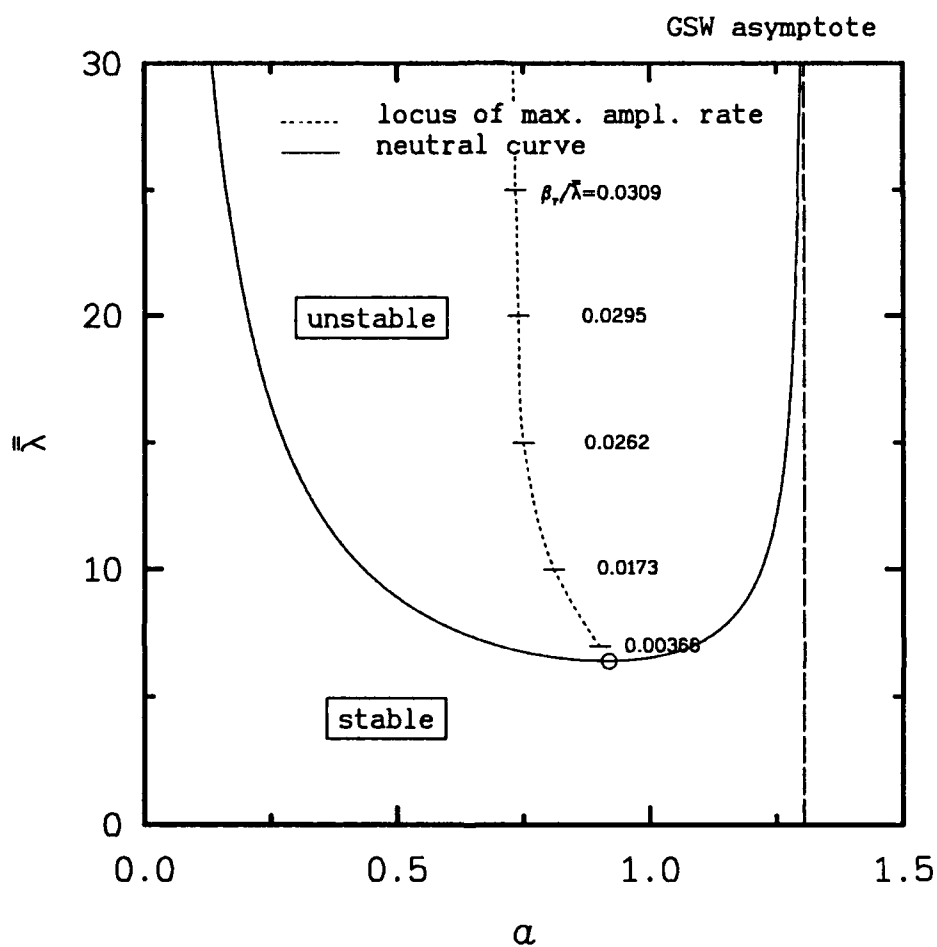


Fig. 2 The neutral curve for the stationary inviscid vortices in the flow over a concave wall. Here  $\bar{\lambda}$  denotes the scaled crossflow parameter, while  $\alpha$  is the dimensionless wavenumber in the spanwise direction. The locus of maximum amplification rate is also indicated by the dotted curve.

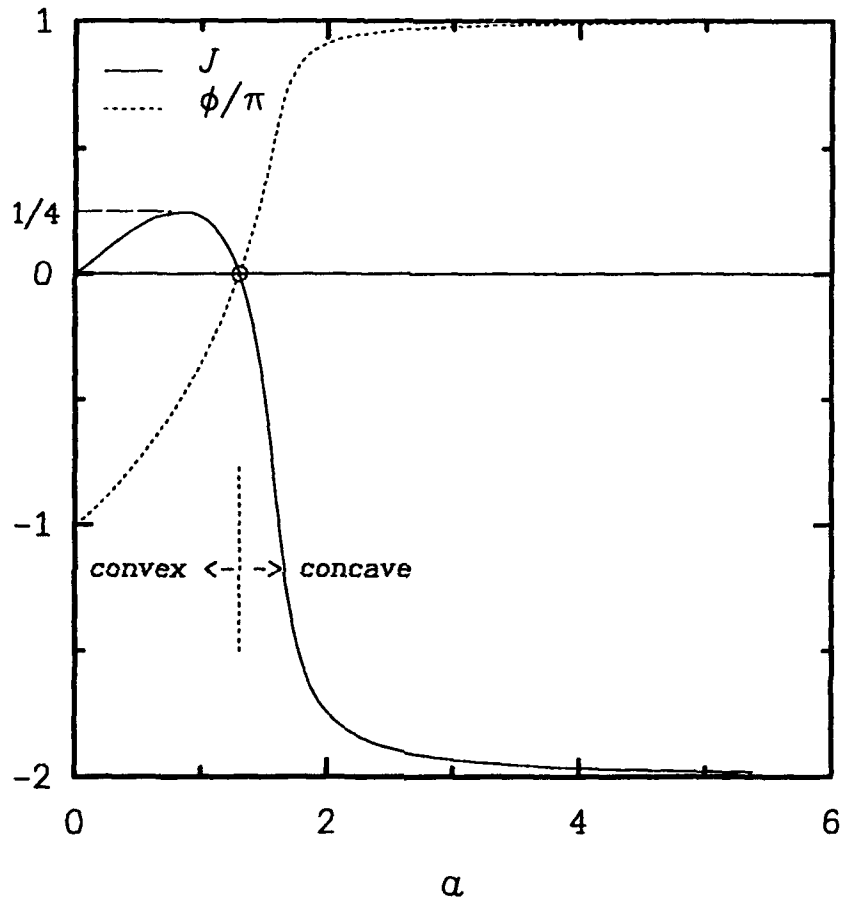


Fig. 3 Phase jump  $\phi$  across the critical layer in the neutral eigenfunction and the corresponding Richardson number  $J$  as functions of the spanwise wavenumber  $\alpha$ . The link between the concave and convex wall curvatures is indicated by the encircled location,  $\alpha=1.305$ .

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13. ABSTRACT (Maximum 200 words) In this report we consider the inviscid instability of three-dimensional boundary-layer flows with a small crossflow over locally concave or convex walls, along with the inviscid instability of stratified shear flows. We show how these two problems are closely related through the forms of their governing equations. A proposed definition of a generalised Richardson number for the neutrally stable inviscid vortex motions is given. Implications of the similarity between the two problems are discussed.				
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